Birgitt Harstad:

Goursat problems of Gyunter type for entire functions in two variables.

1. Introduction.

We shall here treat a global Goursat problem for entire functions that is not covered by the theorems in [1]. We shall prove a global version of the local theorem of N.M. Gyunter,[2]. Our theorem also contains a border case that is not included in the local theorem. We shall use a mixture of the techniques in [1] and [2] in the proof. We shall however only sketch the proof.

2. Preliminaries.

Let $z = (z_1,\dots,z_m) \in \mathbb{C}^m$. By $\alpha = (\alpha_1,\dots,\alpha_m)$ we denote a multi-index with non-negative integers as components. We write

$$z^\alpha = z^{\alpha_1} \dots z_m^{\alpha_m} .$$

2.1 DEFINITION

By an entire function $u \in \mathbb{C}^m$ we mean a function u that is given by a power series

$$u(z) = \Sigma\, a_\alpha z^\alpha$$

which is absolutely convergent for all $z \in \mathbb{C}^m$.

It is easy to prove that this means that u is holomorphic, i.e. u is continuous and differentiable in all of $\mathbb{C}^m$ as a function of the complex variables $z_1,\dots,z_m$.

3. A theorem and a lemma.

3.1 THEOREM

Let a_1 and a_2 be complex constants such that

$$a_1 \cdot a_2 \notin [\tfrac{1}{4}, \infty) ,$$

and let a_3, a_4, a_5 and f be entire functions defined in $\mathbb{C}^2$. Then there exists a unique entire function u such that

$$\frac{\partial^2 u}{\partial z_1 \partial z_2} = a_1 \frac{\partial^2 u}{\partial z_1^2} + a_2 \frac{\partial^2 u}{\partial z_2^2} + a_3 \frac{\partial u}{\partial z_1} + a_4 \frac{\partial u}{\partial z_2} + a_5 u + f$$

and

$$u(0, z_2) = u(z_1, 0) = 0 .$$

If

$$a_1 \cdot a_2 = \frac{1}{4}$$

we have the same conclusion as above if $a_3 = a_4 = a_5 = 0$.

REMARK

The case $a_1 \cdot a_2 = \frac{1}{4}$ has not been treated before, not even locally.

The proof makes use of the following lemma:

3.2 LEMMA

Given a $(n-1) \times (n-1)$ - matrix of the type

$$\begin{pmatrix} -1 & \frac{\lambda}{2} & 0 & 0 & \cdot & \cdot & 0 & 0 \\ \frac{\lambda}{2} & -1 & \frac{\lambda}{2} & 0 & \cdot & \cdot & 0 & 0 \\ 0 & \frac{\lambda}{2} & -1 & \frac{\lambda}{2} & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & -1 & \frac{\lambda}{2} \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \frac{\lambda}{2} & -1 \end{pmatrix} .$$

Let Δ_{n-1} be the determinant of the matrix and let $A_{ik}^{(n-1)}$ be the minor corresponding to the element in the i-th row and the k-th column of the matrix.

If $\lambda^2 \notin [1,\infty)$ it follows that $\Delta_{n-1} \neq 0$ for all $n \geq 2$, and there exists a number $H = H(\lambda)$ independent of n such that

$$\frac{1}{|\Delta_{n-1}|} \sum_{i=1}^{n-1} |A_{ik}^{(n-1)}| \leq H .$$

If $\lambda^2 = 1$ we have $\Delta_{n-1} \neq 0$, and

$$\frac{1}{|\Delta_{n-1}|} \sum_{i=1}^{n-1} |A_{ik}^{(n-1)}| \leq (n-1)^2 .$$

REMARK

The existence of the constant H is proved by Gyunter [2].

Proof of the last part of lemma 3.2.

We get from elementary facts in linear algebra that

$$\Delta_{n-1} = (-1)\Delta_{n-2} - (\tfrac{\lambda}{2})^2 \Delta_{n-3} , \quad n = 3,4,\ldots,$$

$$\Delta_1 = -1, \ \Delta_2 = 1 - (\tfrac{\lambda}{2})^2 . \quad \text{Let } \Delta_0 = 1 .$$

Here $\lambda^2 = 1$ and we have

$$\Delta_{n-1} = n(-\tfrac{1}{2})^{n-1} , \quad n = 2,3,\ldots .$$

Let $A_{ik}^{(n-1)}$ have the same meaning as above. We have that

$$A_{ik}^{(n-1)} = \begin{cases} (-1)^{k+i} \ (\frac{\lambda}{2})^{k-i} \Delta_{i-1} \ \Delta_{n-1-k}, & i = 1,\ldots,k-1 \\ (-1)^{i+k} \ (\frac{\lambda}{2})^{i-k} \Delta_{k-1} \ \Delta_{n-1-i}, & i = k,\ldots,n-1 . \end{cases}$$

Accordingly

$$\frac{1}{|\Delta_{n-1}|}\sum_{i=1}^{n-1}|A_{ik}^{(n-1)}| \le \frac{1}{|\Delta_{n-1}|}\sum_{i=1}^{k-1}(\tfrac{1}{2})^{k-i}|\Delta_{i-1}|\Delta_{n-1-k}|$$

$$+\frac{1}{|\Delta_{n-1}|}\sum_{i=k}^{n-1}(\tfrac{1}{2})^{i-k}|\Delta_{k-1}||\Delta_{n-1-i}| ,$$

or

$$\frac{1}{|\Delta_{n-1}|}\sum_{i=1}^{n-1}|A_{ik}^{(n-1)}| \le \sum_{i=1}^{k-1}(\tfrac{1}{2})^{k-i}\ i(\tfrac{1}{2})^{i-1}(n-k)(\tfrac{1}{2})^{n-1-k}\,\frac{1}{n}(\tfrac{1}{2})^{1-n}$$

$$+\sum_{i=k}^{n-1}(\tfrac{1}{2})^{i-k}\ k(\tfrac{1}{2})^{k-1}(n-i)(\tfrac{1}{2})^{n-1-i}\,\frac{1}{n}(\tfrac{1}{2})^{1-n}$$

$$=\frac{2(n-k)}{n}\sum_{i=1}^{k-1}i+\frac{2k}{n}\sum_{i=k}^{n-1}(n-i)$$

$$=\frac{2(n-k)}{n}\,\frac{k(k-1)}{2}+\frac{2k}{n}\,\frac{(n-k-1)(n-k)}{2}$$

$$=k(n-k)<(n-1)^2$$

since $1 \le k \le n-1$. The lemma is proved.

4. A sketch of the proof of th. 3.1.

We choose new variables such that

$$z_2=\sqrt{\frac{a_2}{a_1}}\,z_2'$$

for $a_1 \neq 0$. (For $a_1 = 0$ see [3].)
Set $\frac{\lambda}{2}=\sqrt{a_1\cdot a_2}$.
We delete the primes and get an equivalent system in the form

$$\frac{\partial^2 u}{\partial z_1 \partial z_2}=\frac{\lambda}{2}\frac{\partial^2 u}{\partial z_1{}^2}+\frac{\lambda}{2}\frac{\partial^2 u}{\partial z_2{}^2}+a_3\frac{\partial u}{\partial z_1}+a_4\frac{\partial u}{\partial z_2}+a_5 u+f ,$$

$u(0,z_2)=u(z_1,0)=0$.

Now we can use lemma 3.2 and the technique in [1] and [2] for the proof of theorem 3.1. This is done by induction as in [1]. The complete proof can be found in [3].

REFERENCES.

[1] Persson, J. : Linear Goursat problems for entire functions when the coefficients are variable. Ann.Sc. Norm.Sup. Pisa Cl.Sc. s.III v. XXIII, 87-99 (1969).

[2] Gyunter, N.M.: On analytic solutions of the equation

$$s = f(x,y,z,p,q,r,t) .$$

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[3] Harstad, B. : Linear Goursat problems for entire functions in two variables with conditions of Gyunter type on the operators (Norwegian). University of Oslo (Thesis) (1970).